COMPUTER CORNER

EDITOR

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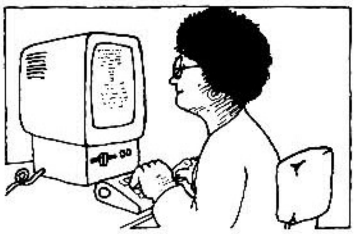

In this column, readers are encouraged to share their expertise and experiences with computers as they relate to college-level mathematics. Articles that illustrate how computers can be used to enhance pedagogy, solve problems, and model real-life situations are especially welcome.

Classroom Computer Capsules features new examples of using the computer to enhance teaching. These short articles demonstrate the use of readily available computing resources to present or elucidate familiar topics in ways that can have an immediate and beneficial effect in the classroom.

Send submissions for both columns to Eugene A. Herman.

Computer-Aided or Analytic Proof?

Hervé Lehning

Hervé Lehning was a student at the Ecole Normale Supérieure of Technology (Paris) until he received his Agrégation in mathematics in 1976. At the same time, he worked as a computer analyst in an insurance company. Since 1981, he has been teaching mathematics to undergraduates at Janson de Sailly (Paris) and computer science to graduates at the Ecole Centrale de Paris. He has written several books and articles on the use of computers in mathematics and its teaching. When he has time, he particularly enjoys rock climbing, mountaineering and a peaceful family life.

A friend who loves mathematical puzzles once asked me, "What is the next number in the sequence 0, 10, 1110, 3110, 132110, 13123110, ... ?"

My answer was 23124110, and the story might have ended there; but, though I do not know why, I continued: 1413223110, 1423224110, 2413323110, 1433223110, 1433223110, and then the sequence is constant from this term on. (Perhaps you have noticed that each term of the sequence is obtained from the previous one by counting the number of occurrences of each digit from 9 down to 0, catenating this count with the digit, and joining these numeric strings to form the new term.) Since this result was surprising, I tried other initial values, and I always found that the resulting sequence was ultimately periodic. A conjecture was thus born. This paper is a statement of some of my thoughts on the matter.

As it happens, my friend's sequence is a generalization of a sequence used by Douglas Hofstadter [2, pp. 389–390] to solve a delightful puzzle invented by Raphael Robinson. The puzzle is to fill in the blanks in the sentence below so that the sentence is true.

> In this sentence, the number of occurrences of 0 is ____, of 1 is ____, of 2 is ____, of 3 is ____, of 4 is ____, of 5 is ____, of 6 is ____, of 7 is ____, of 8 is ____, of 9 is ____.

In studying these peculiar sequences, we shall find the (only two) solutions of Robinson's puzzle. (See Section 4 below. Also see [4] for some related puzzles and their solutions and [3, p. 7, prob. 6] for an interesting related problem.)

1. First Theoretical Study

We shall designate any one of our sequences by the symbol S, and s_n will denote the nth term (or number) of the sequence. The initial term (s_0) may be chosen arbitrarily, after which the terms are formed as follows:

$$s_n = a_9(n)9/a_8(n)8/\cdots/a_0(n)0,$$

where $a_i(n)$ is the number of occurrences of the digit i in s_{n-1}, with the agreement that the ith place is left empty if $a_i(n)=0$. (The $a_i(n)$ will be called the "coefficients" of the term s_n.) Let α_n be the maximum of the $a_i(n)$. Our aim is to show that the α_n are bounded, and to find an economical upper bound for sufficiently large n.

Suppose that α_k has d digits, where $d > 2$. Then each $a_i(k)$ has at most d digits, and there are at most $10d$ digits in all of the $a_i(k)$ together. Even if all of these digits are the same (the worst case for our argument), there can be no more than $10d+1$ occurrences of any given digit in the term s_k ($10d$ among the coefficients, plus one for the indexing digit of the same kind). Thus, no coefficient of s_{k+1} is larger than $10d+1$. Since $10d+1 < 10^{d-1}$ when $d > 2$, it follows that no coefficient of s_{k+1} has more than $d-1$ digits, and, therefore, α_{k+1} has no more than $d-1$ digits.

Repeating the above argument, we see that a term s_m must eventually occur for which α_m has at most two digits. Then each $a_i(m)$ has at most two digits, and there are at most 20 digits in all of the $a_i(m)$ together. Even if all of these digits are the same, there can be no more than 21 occurrences of any given digit in the term s_m. Thus, no coefficient of s_{m+1} is larger than 21. Repetition of the last argument shows that, for all $n > m$, $\alpha_n \leq 21$.

I have demonstrated that for any S there is an N such that all of the α_n for $n \geq N$ are bounded by 21. It follows that, from this point on, the terms $\{s_n\}$ can have at most 22^{10} different values. Eventually, a term must be repeated, and then all subsequent terms will repeat previous ones, since the entire future of S is determined by any single term. Consequently, any S is ultimately periodic, and the period does not exceed 22^{10}. Naturally, we would like to sharpen our estimate of the lengths of possible periods. To this end, let us gather some experimental evidence by computation.

2. Numerical Experimentation

For this we build some software that gives the terms of the sequence and the period, with the input being the initial term. If you would like to do it yourself, I think that you will find it helpful to have a representation of terms that allows easy separation of the digits. One way might be to represent each term as a string of digits. Then you will need a transforming procedure that can inventory the digits of an input term and produce the successor term as output. You will also need a way to store the terms as they are produced; an array (of strings, if you are using a string representation) seems simplest. Now write a program to accept an initial term, compute successive terms, store them, and check whether each new term is equal to some previous one.

When we use such software, we find that, for initial values from 0 through 39, the sequence is ultimately constant. For an initial value of 40, we obtain

$$40, 14/10, 14/21/10, 14/12/31/10, 14/13/12/41/10, 24/13/12/51/10,$$
$$15/14/13/22/41/10, 15/24/13/22/51/10, 25/14/13/32/41/10,$$
$$15/24/23/22/41/10, 15/24/13/42/31/10, 15/24/23/22/41/10,$$

after which the sequence repeats with period two. For an initial value of 50, we find a sequence that is ultimately periodic with period three. Try as hard as we might, we find no (ultimate) periods other than one, two, and three. This fact leads us to conjecture that these are the only possible periods. I shall call this "Lehning's Conjecture," as my wife did at the time when this engrossing sequence disturbed our peaceful life so much.

3. Proof by Computer

To prove this conjecture, we might check each of the finitely many cases resulting from the analysis of Section 1. It would be easy to write a computer program for this purpose, but it would be a mistake to expect it to complete its task in a reasonable amount of time, because 22^{10} is such a large number. (If 10,000 cases could be tested each second, it would take eighty-four years to finish the job!) So we want to reduce the number of cases to be checked. To accomplish this reduction, I shall use the following arguments repeatedly.

Argument 1. Because every sequence S is ultimately periodic, it is sufficient to consider initial terms that belong to cyclic tails. I shall use the word "cycle" to designate a cyclic tail of a sequence, and a cycle of period one will be called a "fixed point."

Argument 2. If a term s_n has its ith coefficient ($a_i(n)$) different from zero, all subsequent terms will also have this coefficient ($a_i(m)$ for $m > n$) different from zero. In words, term generation cannot annihilate any coefficient. In a cycle, term generation also cannot create any new (nonzero) coefficients, since new ones cannot subsequently be destroyed, and the terms repeat.

Argument 3. For any S, $\Sigma_i a_i(n+1)$ is equal to the total number of digits in s_n. I established in Section 1 that, in a cycle, $\alpha_n \leq 21$ for all n; hence, each $a_i(n)$ has no more than two digits, which means that s_n has at most thirty digits. Thus, $\Sigma_i a_i(n+1) \leq 30$ in a cycle. More generally, if we have succeeded in establishing that not more than k coefficients (of any term in a cycle) can be two-digit numbers, with the remaining $10-k$ coefficients being at most one-digit numbers, it then follows that the sum of the coefficients of any term in a cycle is at most $3k+2(10-k)=20+k$.

Argument 4. Suppose we have established that, for some term s_n in a cycle, there are at least r occurrences of each of m different digits among the coefficients $a_i(n)$ of s_n. Then, the sum of the coefficients that contain these digits must be at least $r(1+2+\cdots+m)=rm(m+1)/2$. This is clear if none of the m digits is zero and if no two (of the rm) digits occur in the same $a_i(n)$. If one of the m digits *is* zero, any $a_i(n)$ that contains it must be at least 10, because, by agreement, we leave the ith place blank rather than write "0 i." If two or more (of the rm) digits occur in the same $a_i(n)$, then that $a_i(n)$ is *greater* than the sum of those digits, because of the weight placed upon one (or more) of them by positional notation.

Reducing the number of cases. For any term s_n in a cycle, each $a_i(n)$ has no more than two digits, and $\Sigma_i a_i(n) \leq 30$ (Argument 3).

Suppose that, for some $i \geq 2$, $a_i(n)$ has two digits. Then the digit i occurs at least nine times among the $a_j(n-1)$'s. It cannot occur twice in any $a_j(n-1)$, because if it did that $a_j(n-1)$ would be at least 22, and 22 plus at least 14 more for the remaining seven i's would make the sum of the $a_j(n-1)$'s exceed 30, the upper bound established above. Thus, at least nine different $a_j(n-1)$'s contain the digit i. Let $j_1, j_2, \ldots, j_9$ be the indices of these occurrences. Looking at the term s_{n-2}, we see that each of the digits j_1 to j_9 must occur at least $i-1$ times in the $a_k(n-2)$'s. Thus, the sum of the $a_k(n-2)$'s that contain the $i-1$ copies of j_1 through j_9 must be at least $(2-1)(1+2+\cdots+9)=45$, by Argument 4. However, $\Sigma_k a_k(n-2) \leq 30$ if s_{n-2} is in a cycle. This contradiction shows that, for all s_n in a cycle, and for all $i \geq 2$, $a_i(n)$ is (at most) a one-digit number.

From the fact that the only possible two-digit $a_i(n)$'s are $a_0(n)$ and $a_1(n)$, we conclude that, for s_n in a cycle, $a_0(n) \leq 3$, and, hence, $a_0(n)$ has (at most) one digit. It follows (from Argument 3) that $\Sigma_i a_i(n+1) \leq 21$, and this will be true for all terms in a cycle. Since $a_1(n)$ is the only possible two-digit coefficient, we conclude that (in a cycle), $a_1(n) \leq 12$. The number of cases for computer analysis has been reduced to at most $10^8 \times 13 \times 4$ (10 possibilities for each of a_9 through a_2, 13 possibilities for a_1, and 4 possibilities for a_0). This count will presently be reduced much more, but first I shall analyze the cases for which $a_1(n) > 9$.

The number of 1's. If $a_1(n) > 9$ for some term in a cycle, there are three possibilities.

(1) If $a_1(n)=12$, then s_{n-1} must be 19/18/17/16/15/14/13/12/111/10. The evolution of S is

19/18/17/16/15/14/13/12/121/10, 19/18/17/16/15/14/13/22/111/10,
19/18/17/16/15/14/13/22/111/10,

and from this point on, S is a constant sequence. While we have found a fixed point that will be of subsequent use, we have also shown that $a_1(n)=12$ cannot happen in a cycle.

(2) If $a_1(n) = 11$, let us consider two subcases.

(a) If $a_1(n-1) \neq 11$, then every $a_i(n-1) = 1$ for $i \neq 1$, and $a_1(n-1)$ is either 1 or 10. (The case of $a_1(n-1) = 12$ has been covered by the analysis of the preceding paragraph.) Thus, there are two possibilities for S beginning with the $(n-1)$st term:

19/18/17/16/15/14/13/12/11/10, 19/18/17/16/15/14/13/12/111/10,
19/18/17/16/15/14/13/12/121/10,

leading to the fixed point of (1) above; or

19/18/17/16/15/14/13/12/101/10, 19/18/17/16/15/14/13/12/111/20,
19/18/17/16/15/14/13/22/111/10,

which is again the fixed point of (1).

(b) If $a_1(n-1) = 11$, then exactly eight of the $a_i(n-1)$'s $(i \neq 1)$ are equal to 1. If the other $a_i(n-1)$ is zero (nine cases), the $(n-1)$st term is one of the nine fixed points exemplified by 19/18/16/15/14/13/12/111/10 (with $a_7(n-1) = 0$). If the other $a_i(n-1)$ is not zero, let j be its value, with $j \geq 2$. Then $a_j(n) = 2$, $a_1(n) = 11$, and all other $a_i(n)$'s are equal to 1. Hence, the term s_{n+1} is

19/18/17/16/15/14/13/22/111/10,

which is the fixed point found in (1) above.

(3) If $a_1(n) = 10$, we may suppose that $a_1(n-1) \leq 10$, else we are brought back to one of the previous cases, with n replaced by $n-1$. We can then divide this case into three subcases.

(a) Clearly, $a_1(n-1)$ cannot be zero, so the first subcase to consider is $a_1(n-1) = 1$. Then nine of the $a_i(n-1)$'s (including $a_1(n-1)$) are 1's. By Argument 2, at least nine of the $a_j(n-2)$'s are nonzero; since at most one of them can be 1, eight (or more) of them must be at least 2, in which case at least eight *different* nonzero values appear among the $a_k(n-3)$'s. But, this would imply that $\Sigma_k a_k(n-3) \geq 1 + 2 + \cdots + 8 = 36$, which contradicts the fact that $\Sigma_i a_i(n) \leq 21$ for all terms in a cycle. Thus, this subcase cannot arise.

(b) If $2 \leq a_1(n-1) \leq 9$, then $a_i(n-1) = 1$ for each $i \neq 1$. The term s_n thus has eight a_i's equal to 1, one (a_1) equal to 10, and one (the one whose *index* is $a_1(n-1)$) equal to 2. It follows that s_{n+1} is 19/18/17/16/15/14/13/22/101/20, and the evolution of the sequence is

19/18/17/16/15/14/13/32/91/20, 29/18/17/16/15/14/23/22/81/10,
19/28/17/16/15/14/13/42/71/10, 19/18/27/16/15/24/13/22/81/10,
19/28/17/16/15/14/13/42/71/10,

at which point we have reached a cycle of period two in which all coefficients are smaller than 10. (This cycle will be rediscovered in Section 5.) Since the term s_n is not itself in the cycle, this subcase also cannot arise.

(c) The final case to be considered is the one for which $a_1(n-1) = 10$. We may suppose that $a_1(n-2) = 10$ also, else we would be brought back to one of the previous cases. Since 0 occurs in a coefficient of s_{n-2}, $a_0(n-1)$ is nonzero, whence $a_0(n-2)$ is also nonzero by Argument 2. Thus, $a_0(n-1)$ is at least 2, and it can't be larger than 2 because at most one coefficient of s_{n-2} has more than one digit.

Then s_{n-1} has the form

$$19/18/17/16/15/14/13/12/101/20,$$

and s_n is the same as the term s_{n+1} in subcase (3)(b) above. Thus, the same cycle of period two ensues, and s_n itself is not in the cycle.

Conclusions. Except for the ten fixed points given in cases (1) and (2)(b) above, $a_i(n) \leq 9$ for all i (and all n) if we are in a cycle. From Argument 3, we conclude that $\Sigma_i a_i(n) \leq 20$ for all s_n in a cycle other than these ten fixed points.

Further reduction in the number of cases. From now on, we assume that $a_i(n) \leq 9$ for each i and n, since the cases for which $a_1(n) \geq 10$ have been disposed of in the preceding section. Suppose that, for a given $i \geq 2$, $a_i(n) = k$. Then there are (at least) $k-1$ occurrences of the digit i among the $a_j(n-1)$'s. Let $j_1, j_2, \ldots, j_{k-1}$ be the indices of these occurrences. Looking at the term s_{n-2}, we see that each of the digits j_1 to j_{k-1} must occur at least $i-1$ times among the $a_k(n-2)$'s. Therefore, the sum of the $a_k(n-2)$'s that contain the $i-1$ copies of j_1 through j_{k-1} must be at least $(i-1)(1+2+\cdots+k-1)$ by Argument 4. It follows (from the conclusions of the previous paragraph) that $(i-1)k(k-1)/2 \leq 20$, or, equivalently, $k(k-1) \leq 40/(i-1)$. Therefore, $a_9(n) \leq 2$, $a_8(n) \leq 2$, $a_7(n) \leq 3$, $a_6(n) \leq 3$, $a_5(n) \leq 3$, $a_4(n) \leq 4$, $a_3(n) \leq 5$, and $a_2(n) \leq 6$. By assumption, $a_1(n) \leq 9$. Since zero cannot occur in any coefficient, $a_0(n) \leq 1$ for all cycles that we are now considering.

Our analysis has therefore reduced the number of cases to be tested to $3 \times 3 \times 4 \times 4 \times 4 \times 5 \times 6 \times 7 \times 10 \times 2 = 2{,}419{,}200$. The restriction that $\Sigma_i a_i(n) \leq 20$ further reduces the number of cases to 1,500,043 (by actual count), which is a reasonable number for even a microcomputer to check.

A program to check all cases. A program to test all of these cases needs two components: one to produce the 1,500,043 initial terms of sequences to be tested, and a second to check each sequence. The first component is not difficult to write, so I shall limit my explanations to the second component.

In order to check a sequence S, we commence with s_0 and use the transforming procedure (of our earlier program) a certain number (N) of times in order to reach the cyclic tail of S. Rather than trying to determine the value of N for each S, I have found experimentally that the constant $N = 6$ suffices for all S, provided that certain other tests (to be described presently) are made on candidate sequences. After N transformations, we transform at most three more times, and we check whether s_{N+1}, s_{N+2}, or s_{N+3} is equal to s_N.

One of the tests that should be made after each transformation is that each of the coefficients $a_i(n)$ still conforms to the limits determined in the preceding section. For example, the initial term 19/18/17/16/15/14/13/12/11/10 would transform to a term in which a_1 is 11, and this should not be allowed to happen. Any sequence that leads to an out-of-range term should be discarded.

The other test that should be made is that each transformed term must have the same sum-of-coefficients as the term from which it arose. The reason for this is that the sum of the coefficients is the number of digits in the preceding term, and this number neither expands nor contracts in any cycle in which all coefficients are one-digit numbers. (Refer to Argument 2.)

To make your program run faster, use a simpler representation of terms than was possible in Section 2. An array of eleven bytes is convenient: ten bytes to store the values of a_0 through a_9, and an eleventh byte to hold the sum of these ten coefficients. The fact that all $a_i(n)$'s are one-digit numbers greatly simplifies both the representation of terms and the transformation procedure.

Running the program. When you run the program described above, you may find "Lehning's theorem is true" on the screen of your computer after a half-hour (if your computer is a fast one). What is your opinion of this proof? Is it a *real* proof for you? It is for me!

Once I knew the theorem to be true, I modified the program so that it discarded all initial terms that were not in a cycle, and I added counters to count how many different cycles there are of each of the possible periods. The result was:

99 fixed points
31 cycles of period 2 (each one counted twice, of course)
10 cycles of period 3 (each one counted three times).

The ten fixed points found in the analysis of the subsection "The number of 1's" bring the total number of fixed points to 109. Because there were so few *different* endings of sequences, and just for the pleasure of it, I looked for an analytic proof of Lehning's theorem. In Section 5, I outline my results. Before I do that, I wish to use the results obtained so far to solve Robinson's puzzle.

4. Robinson's Puzzle

Robinson's puzzle (stated in the introduction) is to fill in the blanks in a certain self-inventorying sentence so as to make the sentence true. Hofstadter suggests the following method (which he calls "Robinsonizing") of solving this puzzle. Fill in the blanks with arbitrary numbers to form the initial term of a sequence, and then transform each term into the next by the method defined at the beginning of this paper. If you find a fixed point, that fixed point is a solution (and, conversely, every solution is a fixed point of such a sequence).

In Section 3 we found the fixed point 19/18/17/16/15/14/13/22/111/10 that employs all ten of the digits 0–9. The computer program lists just one other fixed point that uses all ten digits: 19/18/27/16/15/14/23/32/71/10. Thus, these are the only two solutions of Robinson's puzzle. Moreover, the program lists just one cycle of period two that involves all of the digits, and that is the cycle 19/18/27/16/15/24/13/22/81/10, 19/28/17/16/15/14/13/42/71/10. This cycle would solve the puzzle consisting of two sentences similar to the one of Robinson's puzzle, but with each sentence referring to the other. Finally, there is no cycle of period three that involves all of the digits.

5. Outline of an Analytic Proof

From the analysis of Section 3, we may restrict our attention to cycles in which $a_i(n) \le 9$ for each i. We shall need a bit of notation:

$I1(n)$ will represent the set of indices i for which $a_i(n) > 0$. By Argument 2 of the preceding section, $I1(n)$ is independent of n. In what follows, $I1(n)$ will be abbreviated to "I."

$N1$ will represent the number of indices in I. Equivalently, $N1$ is the number of nonzero coefficients of s_n.

$I2(n)$ will represent the set of indices i for which $i \neq 1$ and $a_i(n) > 1$. This set may depend upon n, even in a cycle.

$N2(n)$ will represent the number of indices in $I2(n)$. Equivalently, $N2(n)$ is the number of coefficients (other than $a_1(n)$) greater than 1.

Some basic formulas can now be established. From Argument 3 of the previous section we conclude that

$$\sum_I a_i(n) = 2N1. \tag{1}$$

If i belongs to I, then s_{n-1} has $a_i(n)-1$ indices j for which $a_j(n-1) = i$. Summing the coefficients of s_{n-1} yields $\Sigma_I a_i(n-1) = \Sigma_I i(a_i(n)-1)$. From this last equation, together with (1) and the fact that we are in a cycle, we deduce that $\Sigma_I i(a_i - 1) = 2N1$. From this equation and (1) we get $\Sigma_I (i-2)(a_i-1) = 2N1 - 2\Sigma_I(a_i - 1) = 2N1 - 2(2N1 - N1)$, whence

$$\sum_I (i-2)(a_i - 1) = 0. \tag{2}$$

In (2), the term for which $i = 2$ either does not appear (if $a_2 = 0$) or is zero (because $i - 2 = 0$). The term for which $i = 0$ does not appear (if $a_0 = 0$) or is zero (if $a_0 = 1$), since $a_0 \leq 1$ for the cycles that we are considering. If we assume that $a_1(n) = 0$ for some term in a cycle, then (2) (along with the preceding two sentences) implies that $a_i(n)$ is either 0 or 1 for each $i \geq 3$; $a_i(n) = 1$ for some $i \geq 3$ would make $a_1(n+1) > 0$, in contradiction of Argument 2 of Section 3. Thus, if $a_1(n) = 0$, we are left with the term $a_2(n)2$, from which we deduce (using (1)) that $a_2(n) = 2$, and we find

1 fixed point given by 22.

From now on we suppose that $a_1 > 0$. If $a_1(n) \geq 2$, the number of $a_i(n)$'s equal to 1 is $N1 - N2(n) - 1$, so that $a_1(n+1) = N1 - N2(n)$. Moreover, $2N1 = \Sigma_I a_i(n) = a_1(n) + (N1 - N2(n) - 1) + \Sigma_{I2(n)} a_i(n)$. From these last two equations we conclude:

$$\text{If } a_1(n) \geq 2, \text{ then } a_1(n+1) = a_1(n) - 1 + \sum_{I2(n)} (a_i(n) - 2). \tag{3}$$

Cycles with a constant number of 1's. If a_1 is constant in a cycle, then $a_1 \geq 2$ ($a_1 = 1$ is contradictory), so $a_1(n) = a_1(n+1) = N1 - N2(n)$ by the preceding paragraph, and $N2$ is independent of n. From (3), $\Sigma_{I2(n)}(a_i(n) - 2) = 1$, so one $a_i(n)$ (for $i \neq 1$) is 3 and $(N2 - 1)$ of them are equal to 2. We consider four subcases.

(a) If $a_2(n) = 0$, the digit 2 does not occur, so $N2 = 1$. Then the $a_i(n)$'s are 3, $N1 - 1$ (the coefficient a_1), and $(N1 - 2)$ 1's. If $N1 - 1 \neq 3$, the digit 2 occurs in s_{n+1}, which contradicts our assumption. Thus, $N1 = 4$, and s_{n+1} is of the form $33/31/\langle 2\rangle$, where the symbol $\langle 2\rangle$ represents two substrings of the form $1j$ (with each j differing from all of the displayed indices) embedded somewhere in the string for s_{n+1}. This term is a fixed point, and there are $\binom{8}{2} = 28$ choices for the substring

$\langle 2\rangle$. Thus we have found

$$28 \text{ fixed points of the form } 33/31/\langle 2\rangle.$$

In the other three cases, we assume that a_2 is nonzero.

(b) If $N1 - N2 \geq 4$, then 1, 2, 3 and $N1 - N2$ belong to I. Therefore, s_{n+1} has the form $2(N1-N2)/23/(N2)2/(N1-N2)1/\langle N1-4\rangle$, where the symbol $\langle N1 - 4\rangle$ represents $N1 - 4$ embedded substrings of the form $1j$. The (constant) number of 1's is therefore $N1 - 3$ as well as $N1 - N2$. It follows that $N2 = 3$ and that s_{n+1} is the fixed point $2(N1-3)/23/32/(N1-3)1/\langle N1-4\rangle$. Thus we have

20 fixed points of the form $24/23/32/41/\langle 3\rangle$
15 fixed points of the form $25/23/32/51/\langle 4\rangle$
6 fixed points of the form $26/23/32/61/\langle 5\rangle$
1 fixed point given by $19/18/27/16/15/14/23/32/71/10$.

(c) If $N1 - N2 = 3$, then s_{n+1} has the form $33/(N2)2/31/\langle N1-3\rangle$. It follows that $N2 = 2$ and $N1 = 5$; so, s_{n+1} is one of the

$$21 \text{ fixed points of the form } 33/22/31/\langle 2\rangle.$$

(d) If $N1 - N2 = 2$, then s_{n+1} has the form $23/(N2+1)2/21/\langle N1-3\rangle$. Thus, $N2 = 2$, $N1 = 4$, and s_{n+1} is one of the

$$7 \text{ fixed points of the form } 23/32/21/\langle 1\rangle.$$

We have found altogether 99 fixed points (with all $a_i \leq 9$), and these are the same as the ones found by my computer program.

Cycles with a nonconstant number of 1's. Assume that a_1 is not constant in a cycle, and let n be such that $a_1(n)$ is maximum and $a_1(n+1)$ is not. Since $a_1(n)$ is at least 2, it follows from (3) that $a_1(n+1) \geq a_1(n) - 1$, and, therefore, $a_1(n+1) = a_1(n) - 1$. For this conclusion, the last term of (3) must be 0; so, we see that the $a_i(n)$'s (for $i \neq 1$) that are greater than 1 are all equal to 2. Let $i_1, i_2, \cdots, i_{N2(n)}$ be the i's ($\neq 1$) for which $a_i(n) = 2$. I have shown (just before (3)) that $a_1(n+1) = N1 - N2(n)$; thus, $a_1(n) = N1 - N2(n) + 1$. When we substitute these values in (2), we get the equation $i_1 + i_2 + \cdots + i_{N2(n)} = N1 + N2(n)$, which implies that $N1 + N2(n) \geq 2 + 3 + \cdots + (N2(n)+1) = N2(n)(N2(n)+3)/2$, and, therefore, $N1 - N2(n) + 1 \geq N2(n)(N2(n)-1)/2 + 1$. We consider four cases.

(a) If $N2(n) \geq 3$, the last inequality in the preceding paragraph implies that $N1 - N2(n) + 1$ is at least 4. Thus, 1, 2, and $N1 - N2(n) + 1$ belong to I, and s_{n+1} has the form $2(N1 - N2(n) + 1)/(N2(n)+1)2/(N1 - N2(n))1/\langle N1-3\rangle$. We deduce that $a_1(n+2) = N1 - 2$, and this is not greater than $N1 - N2(n) + 1$, since $a_1(n)$ is maximum. Hence, $N2(n) = 3$, and $N1 \geq 6$. We break this case down into three subcases.

(i) If $N1 \geq 8$, then 4 and $(N1-3)$ belong to I, since $(N1-3)$ is $a_1(n+1)$ and 4 is $a_2(n+1)$. Then s_{n+1} is $2(N1-2)/1(N1-3)/14/42/(N1-3)1/\langle N1-5\rangle$, s_{n+2} is $1(N1-2)/2(N1-3)/24/22/(N1-2)1/\langle N1-5\rangle$, and $s_{n+3} = s_{n+1}$. We have found

10 cycles of period two of the form	26/15/14/42/51/⟨3⟩,
	16/25/24/22/61/⟨3⟩,
5 cycles of period two of the form	27/16/14/42/61/⟨4⟩,
	17/26/24/22/71/⟨4⟩,
1 cycle of period two given by	19/28/17/16/15/14/13/42/71/10,
	19/18/27/16/15/24/13/22/81/10.

(ii) If $N1 = 7$, then s_{n+1} is 25/14/42/41/⟨3⟩. Since the number of 4's in this term is 3, 3 belongs to I. Thus, s_{n+1} is 25/14/13/42/41/⟨2⟩, and, if we trace the evolution of this term, we see that we have found

10 cycles of period three of the form	25/14/13/42/41/⟨2⟩,
	15/34/13/22/51/⟨2⟩,
	25/14/23/22/51/⟨2⟩.

(iii) If $N1 = 6$, s_{n+1} is 24/13/42/31/⟨2⟩, and if we trace the evolution of this term, we see that we have found

15 cycles of period two of the form	24/13/42/31/⟨2⟩,
	24/23/22/41/⟨2⟩,

The 31 cycles of period two and 10 cycles of period three found in the above three subcases are the same as those produced by the computer program.

(b) If $N2(n) = 2$, the facts that $a_1(n) = N1 - N2(n) + 1$ and $a_1(n) \geq 2$ imply that $N1 \geq 3$. If $N1 = 3$, then $a_1(n) = 2$, and all three coefficients of s_n are equal to 2. Then $s_{n+1} = 42/11/\langle 1\rangle$, and $a_1(n+2) = 3$, contradicting the maximality of $a_1(n)$. The case $N2(n) = 2$ and $N1 > 3$ also cannot occur in a cycle with a nonconstant number of 1's. To show this, we look at the following subcases. In each subcase, I display the term s_{n+1} and leave it to you to show that the evolution of this term leads to a cycle such that either s_{n+1} is not in the cycle or the number of 1's is constant. (Also, in every case, the ultimate cycle is one of those that has already been found.)

(i) If $N1 \geq 8$, s_{n+1} is $2(N1-1)/1(N1-2)/1(N1-3)/14/13/32/(N1-2)1/\langle N1-7\rangle$. (That $N1-1$, $N1-2$, 3, 2, and 1 are in I is evident from the previous discussions. The $N1-3$ and the 4 appear at later stages. The coefficients of s_{n+1} are determined by the previous information and the fact that their sum is $2N1$.)

(ii) If $N1 = 7$, s_{n+1} is 26/15/14/13/32/51/⟨1⟩.

(iii) If $N1 = 6$, s_{n+1} is 25/14/13/32/41/⟨1⟩.

(iv) If $N1 = 5$, s_{n+1} is 24/13/32/31/⟨1⟩.

(v) If $N1 = 4$, s_{n+1} is 23/32/21/⟨1⟩.

(c) If $N2(n) = 1$, then $a_1(n) = N1$, and s_{n-1} has $N1 - 1$ coefficients that are equal to 1. The other coefficient of s_{n-1} must be $N1 + 1$, because the sum of the coefficients of any term in a cycle (with all $a_i \leq 9$) is $2N1$. Let j be such that $a_j(n-1) = N1 + 1$. Then, all $N1$ of the coefficients of s_{n-2} are equal to j; since the sum of the coefficients of s_{n-2} is $2N1$ (we are assuming that s_{n-2} is in the cycle, too), it follows that $j = 2$. Then the coefficients of s_{n-3} (also assumed to be in the cycle) are all different and must sum to at least $N1(N1+1)/2$, as well as to $2N1$.

Thus, $N1 \leq 3$. If $N1 = 2$, s_n is 23/21, which is clearly not in a cycle, since s_{n+1} would be a *longer* string. If $N1 = 3$, s_n is $24/31/\langle 1 \rangle$, which also is not in a cycle, since s_{n+1} would again be a longer string. Therefore, the case $N2(n) = 1$ cannot arise.

(d) If $N2(n) = 0$, $a_1(n) = N1 + 1$. Then s_{n-1} has all $(N1)$ of its coefficients equal to 1, contrary to the fact that the sum of its coefficients must be $2N1$ if it is to be in a cycle. Thus, this case also cannot arise.

6. Conclusions from the Two Proofs

Is the analytic proof "better" than the proof by computer? This question does not have an obvious answer. The analytic proof confirms only what I had learned from the computer runs, and, without the computer results, I might not have had the courage to attempt an analytic proof. A conservative mathematician may say that the computer proof is no proof at all, but, then, he or she must also say that the four-color theorem is not really proved (see [**1**]).

What are the criteria for a better proof? We might answer as follows:

(i) A shorter proof is better.

(ii) A proof that requires less knowledge is better.

(iii) A proof that fosters a deeper understanding is better.

The criteria (i) and (ii) might lead us to say that the computer proof is better than the analytic proof, while the criterion (iii) suggests that the analytic proof is the better one. To support the latter view, in the next section I generalize the theorem to other bases.

7. Other Bases

The definition of our sequences is meaningful in any base, b, greater than 1. For example, in base 2, with the initial term 0, we obtain the sequence

0, 10, 11/10, 111/10, 1001/10, 111/110, 1011/10, 1001/100, 111/1000, 1001/110,

and the last term is a fixed point.

Obviously, the computer proof can work in only one base at a time, while the analytic proof has at least a chance of working in *every* base. In fact, all of the arguments of Sections 3 and 5 (with obvious modifications to accommodate the base) are correct in all bases greater than 7. What we need, then, are:

(1) For $b \geq 8$, an argument like that of Section 1 to dispose of all cases in which some a_i's may require three or more digits to represent them.

(2) Computer runs to settle the issue for all $b \leq 7$.

In fact, the argument of Section 1 is valid for any base $b \geq 4$. Merely replace the "10" by "b" (and the "20" and "21" by $2b$ and $2b+1$, respectively), and observe that $bd + 1 < b^{d-1}$ for all $b \geq 4$ and all $d > 2$. If $b \geq 8$, the arguments of Sections 3 and 5 then suffice to establish Lehning's theorem and to determine the forms of all possible cycles.

If $b \leq 7$, it is easy to check all cases with a computer. Arguments similar to those in Section 3 can be used to reduce the number of cases that need to be considered. For example, only 36,548 initial terms need to be generated in base 7. For $b = 2$, 4, or 5, we find only fixed points. For $b = 3$, there are fixed points and one cycle of period three, but no cycle of period two. In base 6, we find fixed points and one cycle of period two. For $b = 7$, we find fixed points, cycles of period two, and one cycle of period three.

Therefore, the theorem is true in any base. All sequences terminate in cycles of periods

one, if $b = 2, 4$, or 5,
one or three, if $b = 3$,
one or two, if $b = 6$, and
one, two, or three, if $b \geq 7$.

As noted above, if $b \geq 8$, the forms of all possible cycles are given in Sections 3 and 5. It is interesting to note that this is also true for bases 4, 5, 6, and 7. The forms of Sections 3 and 5 would predict seven fixed points in base 4, twelve fixed points in base 5, nineteen fixed points and one cycle of period two in base 6, and twenty-nine fixed points, three cycles of period two, and one cycle of period three in base 7, and the computer results coincide exactly with these predictions (including the correct number of each possible form). The only anomalies occur in bases 2 and 3, where the forms of Sections 3 and 5 would predict one fixed point in base 2 and four fixed points in base 3. These fixed points occur, but so do the following anomalous cycles:

1001/110—a fixed point in base 2,
102/21/20, 101/100, 22/101/100—three fixed points in base 3,
and 102/101/10, 12/111/100, 12/121/20—a cycle of period three in base 3.

The theorem is still true for $b = \infty$. In this case, it is sufficient to prove that, for any particular sequence S, the a_i's are bounded, and then to choose a base b greater than this bound.

References

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2. D. Hofstadter, *Metamagical Themas*, Basic Books, New York, 1985.
3. D. Knuth, *The Art of Computer Programming*, Vol. 2, Addison-Wesley, Reading, MA, 1981.
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Do I Have to Work It Out?

Kummer, like all other great mathematicians, was an avid computer, and he was led to his discoveries not by abstract reflection but by accumlated experience of dealing with many specific computational examples. The practice of computation is in rather low repute today, and the idea that computation can be *fun* is rarely spoken aloud.

Harold Edwards, *Fermat's Last Theorem, A Genetic Introduction to Algebraic Number Theory*, Springer-Verlag.