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# DYNAMICS OF TYPICAL CONTINUOUS FUNCTIONS

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ABSTRACT. S. J. Agronsky, A. M. Bruckner, and M. Laczkovic have studied the behaviour of the sequence $(f^n(x))$ where $f$ is the typical continuous function from the closed unit interval $I$ into itself and $x$ the typical point of $I$. In particular, they have proved that the typical limit set $\omega(f, x)$ is a Cantor set of Menger-Uryson dimension zero. Using mainly the Tietze extension theorem, we have found a shorter proof of this result which applies to a more general situation. As a matter of fact, we have replaced the closed unit interval by a compact $N$-dimensional manifold and the Menger-Uryson dimension by the Hausdorff one. We have also proved that, for the typical continuous function $f$, the function $x \to \omega(f, x)$ is continuous at the typical point $x$. It follows that the typical limit set is not a fractal and that, for the typical continuous function $f$, the sequence $(f^n(x))$ is not chaotic.

## INTRODUCTION

From now onward, we set $X$ to denote a compact manifold with boundary, $N$ to denote its dimension, and $d$ to denote a metric on $X$ such that $X$ is a second countable complete metric space (i.e., having a countable basis). If $B$ is a subset of $X$, we shall represent its interior by $\mathrm{Int}(B)$, its closure by $\mathrm{Clos}(B)$, its exterior (i.e., $X - \mathrm{Clos}(B)$) by $\mathrm{Ext}(B)$, its boundary by $\partial B$, and its diameter by $d(B)$.

$K(X)$ will denote the set of compact subsets of $X$ and $d_K$ the Hausdorff metric on $K(X)$ (see [5, p. 96]). Thus, $K(X)$ is a metric space.

$C(X)$ will denote the space of continuous functions from $X$ into itself and $d_\infty$ the metric of uniform convergence on $C(X)$. Thus, $C(X)$ is a complete metric space. With the metric $\delta = \max(d_\infty, d)$, $C(X) \times X$ is also a complete metric space. If $(f, x) \in C(X) \times X$, we shall denote by $\omega(f, x)$ the limit set (i.e., the set of the limit points) of the sequence $(f^n(x))$. Thus $\omega$ is a function from $C(X) \times X$ into $K(X)$.

If $E$ is a complete metric space, we shall call residual a subset of $E$ which contains a countable intersection of dense, open subsets. If $P(x)$ is a property, the sentences "for $x$ typical in $E$, $P(x)$", "if $x$ is typical in $E$, $P(x)$", and "$P(x)$ at the typical point $x$ of $E$" will mean "there is a residual subset $A$ of

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$E$ such that, if $x$ belongs to $A$, then $P(x)$". In the same situation, we may also say that "$P(x)$ is typical in $E$" or "$P(x)$ is typical".

## LEMMAS

The proofs of our theorems rely on two lemmas which mainly use the Tietze extension theorem (in a particular version, see [2, Corollary 1, p. 82]).

**Lemma 1.** *Let $\alpha > 0$, $\beta > 0$, and let $E_{\alpha,\beta}$ be the set of those $(f, x) \in C(X) \times X$ such that there is a finite number of subsets $(k_i)_i$ verifying:*

- *$d(k_i) < \beta$ for each $i$;*
- *$\omega(f, x) \subset \bigcup_i k_i$ and $\sum_i d(k_i)^\alpha < 1$.*

*The set $E_{\alpha,\beta}$ contains an open, dense set in $C(X) \times X$.*

*Proof.* Let $(f, a) \in C(X) \times X$ and $\varepsilon > 0$. Using the compactness of $X$, we construct a finite cover of open sets of diameters strictly less than $\varepsilon$ having a closure homeomorphic to $I^N$ where $I$ is the closed unit interval.

Let $p$ be the smallest positive integer such that there are an integer $n < p$ and an element $V$ of the cover containing $f^p(a)$ and $f^n(a)$. Let $m = p - n$. This property implies that the points $a, f(a), f^2(a), \dots, f^{n+m-1}(a)$ are distinct.

Let $A$ be a closed subset, $B$ and $C$ two open subsets, and $y \notin A$ such that $f(y) \in B$ and $y \in C$. Using the continuity of $f$ at the point $y$, we can construct a compact subset $K$ homeomorphic to $I^N$ such that: $K \cap A = \varnothing$, $K \subset C$, $y \in \operatorname{Int}(K)$, $d(K) < \varepsilon$, and $f(K) \subset B$.

We use this property for $A = \{a, f(a), \dots, f^{n+m-2}(a)\}$, $y = f^{n+m-1}(a)$, $B = V$, and $C = X$, and obtain a compact $K_{n+m-1}$. Using it for $A = \{a, f(a), \dots, f^{n+m-3}(a)\} \cup K_{n+m-1}$, $y = f^{n+m-2}(a)$, $B = \operatorname{Int}(K_{n+m-2})$, and $C = X$, we obtain $K_{n+m-2}$. We continue in the same way (with the exception of $C = V$ for $K_n$). Eventually, we have constructed a sequence of compacts $K_0, K_1, \dots, K_{n+m-1}$ homeomorphic to $I^N$ such that:

- $a \in \operatorname{Int}(K_0)$;
- $d(K_i) < \varepsilon$ and $f(K_i) \subset \operatorname{Int}(K_{i+1})$ for $0 \le i \le n + m - 2$;
- $K_i \cap K_j = \varnothing$ for $i \neq j$;
- $K_n \subset V$ and $f(K_{n+m-1}) \subset V$.

Let $\rho > 0$ be small enough that $m\rho^\alpha < 1$ and $\rho < \beta$. For each $i$, we choose a compact $k_i$ with a non-empty interior included in $\operatorname{Int}(K_i)$ such that $d(k_i) < \rho$ and a point $b_i$ in $\operatorname{Int}(k_i)$.

For each $i$, let $\eta_i > 0$ be such that the ball centred at $b_i$ of radius $\eta_i$ is included in $\operatorname{Int}(k_i)$. Then, we define a function $g$ in the following way:

For $0 \le i \le n + m - 2$, we put $g = b_{i+1}$ on $k_i$ and $g = f$ on $\partial K_i$. On the one hand, $g$ takes its values in $K_{i+1}$ which is homeomorphic to $I^N$. On the other hand, $k_i \cup \partial K_i$ is a closed subset of $K_i$. So, according to the Tietze extension theorem (see [2, Corollary 1, p. 82]), $g$ has a continuous extension (which means that it still agrees with $f$ on $\partial K_i$ and with $b_{i+1}$ on $k_i$) defined on $K_i$ and taking its values in $K_{i+1}$. As $d(K_{i+1}) < \varepsilon$, $d[f(x), g(x)] < \varepsilon$ for each $x \in K_i$.

In the same way, we put $g = b_n$ on $k_{n+m-1}$ and $g = f$ on $\partial K_{n+m-1}$, and then we extend $g$ on $K_{n+m-1}$ into $\operatorname{Clos}(V)$.

We put $g = f$ outside $\bigcup_i K_i$.

So, as $g$ agrees with $f$ on the boundaries of the $K_i$, it follows that $g$ is continuous on $X$. As $d[f(x), g(x)] < \varepsilon$ for each $x \in X$ and the function $x \to d[f(x), g(x)]$ is continuous on the compact $X$, we have $d_\infty(f, g) < \varepsilon$.

Let $b = b_0$ and $\eta = \min \eta_i$. As $a$ and $b$ are both in $K_0$, $d(a, b) < \varepsilon$. Thus, we have defined $(g, b) \in C(X) \times X$ such that $\delta[(f, a), (g, b)] < \varepsilon$.

Let $(h, c) \in C(X) \times X$ be such that $\delta[(g, b), (h, c)] < \eta$, that is to say, $d_\infty(h, g) < \eta$ and $d(c, b) < \eta$.

If $x \in k_i$, then $g(x) = b_{i+1}$, so $d[b_{i+1}, h(x)] < \eta$; thus $d[b_{i+1}, h(x)] < \eta_{i+1}$, and so $h(x) \in k_{i+1}$. Therefore, $h(k_i) \subset k_{i+1}$ for each $0 \leq i \leq n+m-2$. For the same reason, $h(k_{n+m-1}) \subset k_n$.

As $c \in k_0$, for $l \geq n$, the sequence $(h^l(c))$ takes its values in $k_n \cup k_{n+1} \cup \cdots \cup k_{n+m-1}$ which is a closed subset. Thus $\omega(h, c) \subset k_n \cup k_{n+1} \cup \cdots \cup k_{n+m-1}$. As $m\rho^\alpha < 1$ and $\rho < \beta$, $d(k_i) < \beta$ for each $i$ and $\sum_i d(k_i)^\alpha < 1$. Therefore, $(h, c)$ belongs to $E_{\alpha, \beta}$. So the ball of $C(X) \times X$ centred at $(g, b)$ of radius $\eta$ is included in $E_{\alpha, \beta}$.

So any ball of $C(X) \times X$ (centred at $(f, a)$ of radius $\varepsilon$) contains a point $(g, b)$ which is the centre of a ball (of radius $\eta$) included in $E_{\alpha, \beta}$, and the result follows.

**Lemma 2.** *Let $U$ be an open set of $X$. Let $F_U$ be the set of those $(f, x) \in C(X) \times X$ such that $\omega(f, x)$ either contains no point of $U$ or contains at least two points of $U$. Then $F_U$ contains an open, dense subset of $C(X) \times X$.*

*Proof.* Let $(f, a) \in C(X) \times X$ and $\varepsilon > 0$. We begin the construction as in the proof of Lemma 1. We obtain $n$ and $m$ as before. We shall distinguish two cases:

(1) For each $i$ such that $n \leq i \leq n+m-1$, $f^i(a) \in \mathrm{Ext}(U)$.

As $\mathrm{Ext}(U)$ is open, we can choose the $K_i$ for $0 \leq i \leq n+m-1$ such that $K_i \subset \mathrm{Ext}(U)$ for $n \leq i \leq n+m-1$. We choose any $\rho > 0$ and then the $k_i$ and the $b_i$. As in the proof of Lemma 1, we obtain $(g, b) \in C(X) \times X$ and $\eta > 0$ such that $\delta[(f, a), (g, b)] < \varepsilon$; and if $(h, c) \in C(X) \times X$ verifies $\delta[(g, b), (h, c)] < \eta$, then $\omega(h, c)$ has no point in $U$, so $(h, c)$ belongs to $F_U$.

(2) There is a $j$ such that $n \leq j \leq n+m-1$ and $f^j(a) \in \mathrm{Clos}(U)$.

We can choose the $K_i$ such that $K_j \cap U$ contains an open set $W$. Taking any $\rho > 0$, we chose the $k_i$ and the $b_i$ as in the proof of Lemma 1 for $i < n$. Then, for each $i \geq n$, we choose $k_i$ and $k'_i$ two disjoint compact subsets with non-empty interiors included in $\mathrm{Int}(K_i)$, $b_i$ and $b'_i$ as in the proof of Lemma 1, and $k_j$ and $k'_j$ in $W$. Then we modify the construction of the function $g$ in the following way:

- for $0 \leq i \leq n+m-2$, we put $g = b_{i+1}$ on $k_i$ and $g = b'_{i+1}$ on $k'_i$;
- we put $g = b'_n$ on $k_{n+m-1}$ and $g = b_n$ on $k'_{n+m-1}$.

As in the proof of Lemma 1, we obtain $(g, b) \in C(X) \times X$ and $\eta > 0$ such that $\delta[(f, a), (g, b)] < \varepsilon$; and if $(h, c) \in C(X) \times X$ verifies $\delta[(g, b), (h, c)] < \eta$, then $\omega(h, c)$ has at least two points in $U$, so $(h, c)$ belongs to $F_U$. The result follows.

## RESULTS

**Theorem 1.** *For $(f, x)$ typical in $C(X) \times X$, $\omega(f, x)$ is a perfect set of Hausdorff dimension zero.*

*Proof*. Let $E$ be the intersection of the $E_{1/n, 1/m}$ for $n, m \geq 1$ and $F_{U_p}$, $(U_p)_p$ being a countable base of open subsets of $X$. As the set of the triples $(n, m, p)$ is countable and according to Lemmas 1 and 2, $E$ is residual.

Let $(f, x) \in E$. For each $(\alpha, \beta)$, $(f, x) \in E_{\alpha, \beta}$; thus, $\omega(f, x)$ can be covered by a finite number of sets $(k_i)_i$ satisfying: for each $i$, $d(k_i) < \beta$ and $\sum_i d(k_i)^\alpha < 1$. Therefore, according to the definition of the Hausdorff dimension (see [2]), $\omega(f, x)$ is of Hausdorff dimension zero.

For each open subset $U$ of $X$, $(f, x) \in F_U$; thus either $\omega(f, x)$ has no point in $U$ or at least two. Therefore, $\omega(f, x)$ does not contain an isolated point. As $\omega(f, x)$ is compact, $\omega(f, x)$ is a perfect set.

So we have found $E$, a residual subset of $C(X) \times X$, such that if $(f, x) \in E$, then $\omega(f, x)$ is a perfect set of Hausdorff dimension zero. That is to say, for $(f, x)$ typical in $C(X) \times X$, $\omega(f, x)$ is a perfect set of Hausdorff dimension zero.

**Theorem 2.** *If $f$ is typical in $C(X)$, then, for $x$ typical in $X$, $\omega(f, x)$ is a perfect set of Hausdorff dimension zero.*

*Proof*. $C(X)$ and $X$ are two complete metric spaces and $X$ is second countable. Let $E$ be the residual subset of $C(X) \times X$ of the proof of Theorem 1. According to the Kuratowski-Ulam theorem (see [4, p. 56]), there is a residual subset $F$ of $C(X)$ such that for each $f \in F$ the section of $E$ by $f$ (i.e., $E_f = \{x \in X; (f, x) \in E\}$) is a residual subset of $X$. If $f \in F$ and $x \in E_f$, then $(f, x) \in E$, so $\omega(f, x)$ is a perfect set of Hausdorff dimension zero. That is to say, if $f$ is typical in $C(X)$, then, for $x$ typical in $X$, $\omega(f, x)$ is a perfect set of Hausdorff dimension zero.

**Theorem 3.** *The function $\omega$ is continuous at the typical point $(f, x)$ of $C(X) \times X$.*

*Proof*. Let $\alpha > 0$ and $G_\alpha$ be the set of those $(f, x) \in C(X) \times X$ such that there is a neighbourhood $U$ of $(f, x)$ on which $d_K[\omega(f, x), \omega(f', x')] < \alpha$ for each $(f', x') \in U$.

Let $(f, a) \in C(X) \times X$ and $\varepsilon > 0$. We begin as in the proof of Lemma 1. We obtain the $K_i$ and then choose $\rho > 0$ such that $m\rho < \alpha$. Then we obtain the $k_i$ such that $d(k_n) + d(k_{n+1}) + \cdots + d(k_{n+m-1}) < \alpha$. Then we construct $(g, b) \in C(X) \times X$ and $\eta > 0$ as in the proof of Lemma 1. Let $B$ be the open ball of $C(X) \times X$ centred at $(g, b)$ of radius $\eta$. If $(h, c) \in B$, then $\omega(h, c) \subset k_n \cup k_{n+1} \cup \cdots \cup k_{n+m-1}$ and $\omega(h, c) \cap k_i \neq \varnothing$ for $n \leq i \leq n+m-1$. Thus, if $(h, c)$ and $(h', c')$ belong to $B$, then $d_K[\omega(h, c), \omega(h', c')] < \alpha$.

Let $(h, c) \in B$. Since $B$ is an open set, there is an open neighbourhood $V$ of $(h, c)$ contained in $B$. If $(h', c') \in V$, then $d_K[\omega(h, c), \omega(h', c')] < \alpha$. Hence $(h, c) \in G_\alpha$. As in the proof of Lemma 1, it follows that $G_\alpha$ contains a dense, open subset of $C(X) \times X$.

The intersection $G$ of the $G_{1/n}$ for $n \geq 1$ is residual. Let $(f, x) \in G$. Let $\alpha > 0$; then $(f, x) \in G_\alpha$, so there is a neighbourhood $U$ of $(f, x)$ such that $d_K[\omega(f, x), \omega(f', x')] < \alpha$ for each $(f', x') \in U$. This means that $\omega$ is continuous at the point $(f, x)$. Therefore, $\omega$ is continuous at any point of $G$. The result follows.

**Theorem 4.** *If $f$ is typical in $C(X)$, the function $x \to \omega(f, x)$ is continuous at the typical point $x$ of $X$.*

*Proof*. As in the proof of Theorem 2, the Kuratowski-Ulam theorem allows us to deduce this result from the previous one.

## REMARKS

*Remark* 1. According to Mandelbrot (see [3, p. 15]), a fractal is a set for which the two dimensions used in this paper (and in [2]) are distinct. Hence the typical limit set is not a fractal.

*Remark* 2. In chaotic sequences, the limit set has a sensitive dependence on the initial condition. As for $f$ typical in $C(X)$, the function $x \to \omega(f, x)$ is continuous at the typical point of $X$; it follows that the sequence $(f^n(x))$ is not chaotic.

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