has explicit geometric meaning. (4) The entire argument can be summarized by the statement, "Apply the Pythagorean Theorem to the right triangle of the figure."

Our proof is essentially the proof given without motivation or interpretation in [**1**, p. 61].

REFERENCE

1. Marvin Marcus and Henryk Minc, A Survey of Matrix Theory and Matrix Inequalities, Prindle, Weber & Schmidt, Boston, 1964.

From Experimentation to Proof

HERVÉ LEHNING
13 rue Letellier, 75015 Paris, France

I particularly took notice of the following problem, proposed by A. Tissier in the MONTHLY [**1**]:

"Prove that the differential equation $y' = x - 1/y$ has a unique solution on $[0, \infty)$ which is positive throughout and tends to zero at $+\infty$."

Before the appearance of personal computers, this kind of problem was hard to assign to students because they found it difficult to see the family of solution curves of a differential equation without solving it by quadratures. In this paper, I would like to use this example to demonstrate an intuitive and experimental approach that not only points to the result but also gives some ideas for the proof.

1. Research. The first idea is to sketch the family of curves by using adapted software (see, for example [**2**]). More precisely, we sketch the solution curves of initial value problems:

$$y' = x - \frac{1}{y}, \qquad y(0) = y_0 > 0.$$

By trying a number of values for y_0, we quickly notice that if y_0 is too large, the solution does not seem to be bounded and if y_0 is too small, it does not seem to be defined in all of $[0, \infty)$. More precisely, our successive trials are: 1, 2, 1.5, 1.2, 1.3, and 1.25. You can see the result in FIGURE 1.

Intuitively, Tissier's result seems correct; moreover the solution seems to be the limit of two sequences; the first one is increasing and the other one is decreasing. More precisely, we can define (in fact, for the time being, it is just a conjecture; we shall have to prove it) two sequences of functions f_n and g_n (see FIGURE 2) that seem to converge to the solution we are looking for.

We also notice that a curve (which is $y = 1/x$ according to the study of the sign of $y' = x - 1/y$) divides the first quadrant into regions where the solution is increasing and decreasing.

See [**3**, **4**, **5**] for other examples of this kind of approach.

2. From research to proof. In fact, if we prove that these sequences are well defined, it is easy to show that f_n is decreasing and g_n increasing (uniqueness of the solution of the initial value problem); and then: $|f_n - g_n| \leqslant 1/n$ because $f_n - g_n$ is

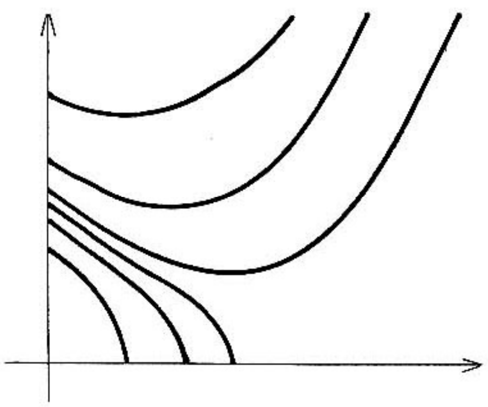

FIG. 1

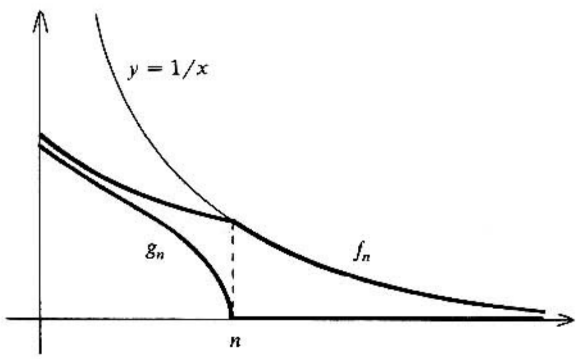

FIG. 2

decreasing in $[0, n]$. We easily deduce that f_n and g_n converge uniformly to the same function f. This function is continuous, so by writing the equation in integral form, we can prove that it is a solution of the problem; what follows is easy because if y is a solution of the equation in $[0, \infty)$:

—if $y(0) > f(0)$ then $y(0) > f_n(0)$ for an integer n and so according to the uniqueness theorem, $y > f_n$ for each x so y is not bounded.
—if $y(0) < f(0)$ then, for the same reason, y is not defined in the whole interval.

In order to remain close to the main subject that we would like to stress, we shall not discuss here the search for the proof of the definition of f_n and g_n because it is a classical question. As a matter of fact, it is sufficient to study the solutions of the initial value problem in the open set defined by the inequalities $y > 0$, $xy < 1$, which is easy, as these solutions are necessarily decreasing (nevertheless, please note that in this case too, the experimental approach is useful).

3. Subtlety. At this step, we can begin proving. Nevertheless, we see that the correct writing of the proof remains awkward since we must introduce some questions of uniform convergence. The same idea can be used in order to obtain a more elementary proof. To each $x > 0$, we assign $f(x)$ and $g(x)$ as shown in FIGURE 3.

Then we show that f is strictly decreasing (because there is a unique solution curve passing through each point, see FIGURE 4) and positive, g is strictly increasing and these two functions satisfy

$$0 \leqslant f(x) - g(x) \leqslant 1/x \quad \text{for each } x.$$

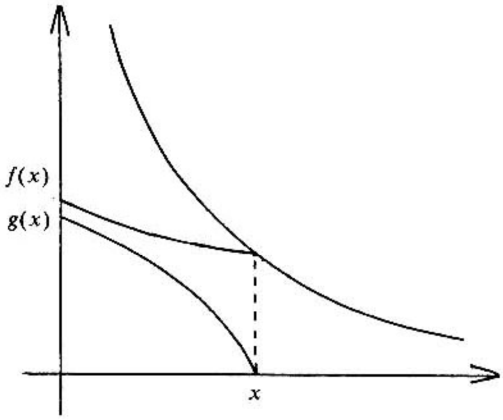

FIG. 3

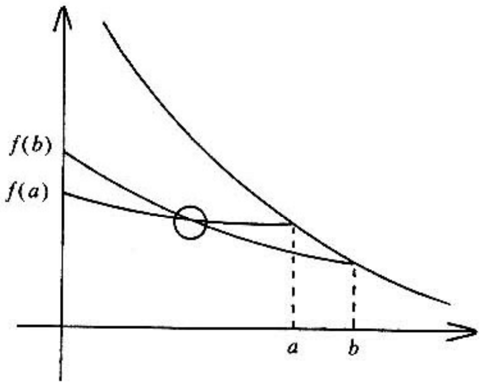

FIG. 4

This allows us to conclude that f and g have a common limit l as x tends to infinity. Then it is easy to show that the solution of the initial value problem at $(0, l)$ is the solution we are looking for. Once these two often hidden, but essential, steps are completed, we can begin to write the proof in its usual baldness. Here, we shall present it in details in order to show that our method does not ruin the notion of rigor.

4. Proof

4.1. First step: Generalities on the solutions of the equation.

(i) Let U be the open set of the plane defined by the inequalities: $y > 0$, $xy < 1$. The function $(x, y) \mapsto x - 1/y$ is continuously differentiable in U so if (x_0, y_0) belongs to U then the initial value problem:

$$y' = x - \frac{1}{y}, \qquad y(x_0) = y_0, \qquad (x, y) \text{ in } U$$

has a unique maximal solution defined in an open interval (x_1, x_2) where x_1 and x_2 can be infinite. Because $x - 1/y < 0$, y is decreasing on (x_1, x_2), so: $y \geqslant y_0$ in $(x_1, x_0]$ and then: $y' \geqslant x - 1/y_0$ in $(x_1, x_0]$. Integrating yields

$$y_0 - y \geqslant \frac{1}{2}\left[\left(x_0 - \frac{1}{y_0}\right)^2 - \left(x - \frac{1}{y_0}\right)^2\right], \quad \text{whence}$$

$$y \leqslant y_0 + (x_0 - x)\left[\frac{1}{y_0} - \frac{x + x_0}{2}\right] \text{ in } (x_1, x_0].$$

If x_1 is finite, we deduce that y has a limit y_1 as x tends to x_1. The solution y being maximal, (x_1, y_1) does not belong to U so either $y_1 = 0$ or $x_1 y_1 = 1$. But these two cases must be excluded because on the one hand $y_1 \geqslant y_0$ and on the other hand, if $x_1 y_1 = 1$, then $(y - y_1)/(x - x_1)$ tends to 0 as x tends to x_1. Now

$$\frac{y - y_1}{x - x_1} \leqslant \frac{\frac{1}{x} - \frac{1}{x_1}}{x - x_1},$$

which leads to a contradiction when passing to the limit. Then $x_1 = -\infty$.

In the same way, as x tends to x_2, y has a finite limit y_2 since it is decreasing and positive. If x_2 is infinite then $y_2 = 0$ since $xy < 1$; if x_2 is finite then either $y_2 = 0$ or $x_2 y_2 = 1$.

So, we have proved that if (x_0, y_0) belongs to U then the initial value problem

$$y' = x - \frac{1}{y}, \qquad y(x_0) = y_0, \qquad (x, y) \text{ in } U$$

has a unique maximal solution y defined in an open interval $(-\infty, a)$ where a is finite or infinite and y tends to $1/a$ or to 0 as x tends to a.

(ii) Let V be the open set defined by the inequalities: $y > 0$, $xy > 1$, and (x_0, y_0) belong to V. In the same way, the initial value problem

$$y' = x - \frac{1}{y}, \qquad y(x_0) = y_0, \quad (x, y) \text{ in } V$$

has a unique maximal solution y defined in an open interval (a, ∞) where $a > 0$; moreover, y tends to $1/a$ as x tends to a and to ∞ as x tends to ∞.

(iii) For $a > 0$, the initial value problem

$$y' = x - \frac{1}{y}, \qquad y(a) = \frac{1}{a}, \qquad y > 0$$

has a unique maximal solution y. The derivative at point a of the function $x \mapsto xy(x)$ is $1/a > 0$ so there are points $x_0 < a$ and $x_1 > a$ that belong to the interval of definition of y and such that $(x_0, y(x_0))$ belongs to U and $(x_1, y(x_1))$ to V. Applying the results obtained in (i) and (ii) at these points, we show that y is defined in $(-\infty, \infty)$.

(iv) So let W be the open set defined by the inequality: $y > 0$. If (x_0, y_0) belongs to W, then the initial value problem $y' = x - 1/y$, $y(x_0) = y_0$, $y > 0$ has a unique maximal solution y defined in $(-\infty, a)$ where a is finite or infinite. If a is finite or if a is infinite and the graph does not intersect the curve whose equation is $xy = 1$, the limit of y at a is 0; otherwise it is $+\infty$.

If $y_{1_0} < y_{2_0}$ (for the same x_0) then the two associated solutions verify $y_1 < y_2$ in the common interval of definition; otherwise, according to the intermediate value theorem, there would be an x such that $y_1(x) = y_2(x)$ and so $y_1 = y_2$ according to the uniqueness theorem, which is a contradiction.

4.2. Second step: definition of functions f and g. Let $a > 0$ and y_a the solution in $(-\infty, \infty)$ of the initial value problem $y' = x - 1/y$, $y(a) = 1/a$, $y > 0$. We define $f(a) = y_a(0)$. Then f maps $(0, \infty)$ into itself. Let $a < b$. Then $y_b(a) < y_a(a) = 1/a$ since $(a, y_b(a))$ belongs to U. Hence $y_b < y_a$ so $f(a) > f(b)$. We deduce that f is strictly decreasing. In the same way, in the open set defined by $xy - 1 < 0$, the initial value problem

$$x' = \frac{y}{xy-1}, \qquad x(0) = a$$

has a unique maximal solution x_a defined in an interval containing $y_0 > 0$. The derivative x_a' does not vanish anywhere in $(0, y_0)$ because $y/(xy-1) < 0$. Thus, x_a has an inverse function z which is the solution of the equation $y' = x - 1/y$ in $[x_a(y_0), a)$. According to the uniqueness theorem, the maximal solution z_a of the initial value problem $y' = x - 1/y$, $y(x_a(0)) = y_0$, (x, y) in U equals z in this interval. So z_a is defined in $[0, a)$ and tends to 0 as x tends to a. Let $g(a) = z_a(0)$. In the same way, we show that g maps $[0, \infty)$ into itself and is strictly increasing, and that $f(x) > g(x)$ for each x. Moreover,

$$y_a' - z_a' = \left(x - \frac{1}{y_a}\right) - \left(x - \frac{1}{z_a}\right) = \frac{y_a - z_a}{y_a z_a}.$$

Then $y_a' - z_a' > 0$ and so $f(a) - g(a) < 1/a$.

4.3. Third step: conclusion. The function f is decreasing and positive, so it converges to a limit l. Let y be the maximal solution of the initial value problem

$$y' = x - \frac{1}{y}, \qquad y(0) = l, \quad y > 0;$$

y is defined in $(-\infty, \infty)$ since otherwise it would be one of the above functions z_a (see first step in (iv)). In the same way, for each x, $(x, y(x))$ belongs to U, since otherwise it would be one of the functions y_a, so y is positive and tends to zero as x tends to ∞. It is the unique solution of our problem because if y is another one, let $b = y(0)$.

If $b > l$, then there is an a such that $b > f(a)$ and then $y > y_a$, so it is not bounded.

If $b < l$, then there is an a such that $b < g(a)$ and then $y < z_a$, so y is not defined beyond a.

5. Postscript. I described this small example in such detail because I think we do not distinguish enough between research and proofs in our teaching. Of course, we all know that mathematical activity requires an experimental phase. But, too often, we imitate Bourbaki's way of writing [6], which hides the research and the trial phases completely. So, some students see mathematics as a dead science whereas mathematics is very much alive.

REFERENCES

1. A. Tissier, problem 6551, this MONTHLY, 94 (1987) 694.
2. Atelier Logiciel de l'Ecole Centrale, Etude des suites, Fil, Paris, 1987.
3. Michèle Artigue, Systèmes Différentiels, Cedic, Paris, 1983.
4. Hervé Lehning, Computers as mathematical problem solving assistants, in A Computer for Each Student, R. Lewis and E. D. Tagg, eds., Elsevier/North Holland, Amsterdam, 1987.
5. ______, Analyse Fonctionnelle, Masson, Paris, 1988.
6. Nicolas Bourbaki, Eléments de Mathématiques, Masson, Paris, 1939–1988.